PENROSE AND THE INDIFFERENT CROWD

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ABSTRACT

Today Lionel Penrose is recognised as the co-author of one of the two leading indices of power in voting legislatures—a field of study that game theory in general, and cooperative game theory in particular, has been reclaiming from sociology and political science since the 1950s. The main claim of this paper is that Penrose developed his index so as to tackle questions that go vastly beyond the narrow domain of voting. Namely, acute social issues during the Cold War such as the outburst and propagation of panics, the ideological susceptibility of populations, the escalation of military conflict and the successful installation of authoritarian regimes. Furthermore, by revisiting the history of the Penrose power index, the paper re-evaluates some of its key underlying assumptions: assumptions that have been heavily—and unfairly, as the paper argues—criticised over the last decade.

Keywords: Lionel Penrose; voting power; social control; Penrose-Banzhaf index; Shapley-Shubik index; cooperative game theory

Acknowledgements: Thanks to the editors of *RHETM*, Steven G. Medema, the participants at the 2018 History of Recent Social Science conference in Zurich, the members of the OZSW Peer-review Circle Practical Philosophy, and the History of Philosophy research group at the VU, Amsterdam. Special thanks to Matthew Braham, Stefan Napel, Mark Solovey, Timothy Barnard and Luca Fiorito.

> The clear lesson here is that consistent and unwavering people, in the public or private sector, can move groups and practices in their preferred direction.
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> —Thaler and Sunstein (2009, p. 62)

> The important statistical fact which emerges from these considerations is the very high degree of control exercised by a comparatively small resolute group over an indifferent population of large dimensions.
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> —Penrose (1952, p. 8)

In July 1962, Lionel Sharples Penrose (1898–1972) had just completed a manuscript on the spread of pathological ideas in human populations. Although tangential to his main area of expertise, such a text would not have been surprising to those familiar with Penrose's work. Three years later, Penrose would retire from the Galton Professorship of Human Genetics at University College London (UCL) as a world authority on human genetics—by that time, he had held this university position for twenty years. Penrose's life-long work had been on the determinants of mental deficiency and, in 1964, his ground-breaking contributions—starting from the Colchester survey, a pioneering large-scale data collection study on the inheritance of mental deficiency—had just been recognised by the Joseph P. Kennedy, Jr. Foundation annual prize awarded by none other than President Lyndon B. Johnson.[1]

[1] Had John F. Kennedy not been assassinated on 22 November 1963, he would have headed the ceremony instead of Johnson (MacDonald, 1972).

And yet that 1962 manuscript on the spread of pathological ideas would be surprising to those today who are aware of Penrose from his non-medical work. Outside the field of human genetics—and, arguably, the artistic sphere[2]—Lionel Penrose is mostly known as the author of the so-called Penrose square root law and, especially, the Penrose voting power index (1946). The Penrose square root law answers the following question: how should a decision-making body, such as the European Commission or the United States Congress, allocate votes among its representatives so that the vote of each represented *citizen* is equally powerful; or, put differently, so that the

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For more on Penrose's life, see Smith (1999)—a short self-published biography of Penrose and the only monograph-length review of all of his wide-ranging interests. Other illuminating accounts are the obituaries by Harry Harris (1974), Penrose's successor at the Galton Laboratory, and Norman MacDonald (1972), Penrose's successor at the head of the Medical Association for the Prevention of War. Cheryl Misak's (2020) recent biography of Frank Ramsey has Penrose as a recurring supporting character and, while not focused on Penrose's work, it provides a rare glimpse into the milieu of his student days.

[2] Together with his son, the mathematician and physicist Roger Penrose, Lionel Penrose is well known for depictions and wooden models of the Penrose or impossible staircase (Penrose & Penrose, 1958): a never-ending staircase loop that inspired M. C. Escher's most famous lithograph, *Ascending and Descending*. These collaborative pieces were part of a series of wooden models, often drawing on Penrose's own medical work (Smith, 1999, pp. 42–47). In 1958, Penrose published a paper on self-reproducing machines which was to aid understanding of DNA replication: the inspiration for the models described in the article was credited to the work on self-reproducing automata of one of the fathers of game theory, John von Neumann (1951).

respective voting institution satisfies the principle of 'one person, one vote'? Representatives should be given voting weights that are proportional to the square root of the district populations which elect them, says the Penrose square root law. The Penrose voting power index, on the other hand, sharpens the meaning of 'the power of a vote'. According to the Penrose index, a voter's power is her probability of being decisive; that is to say, her ability of making a difference to the outcome of a vote.

These two ideas—the square root law and the voting power index—have firmly fixed Penrose's place in the social sciences today: his contributions are treated as contributions strictly to the theory of voting and the design of representative institutions. His square root law is standard fare in the literature on *fair* representation while his power index is one of the two classic measures in the theory of measuring voting power, the other being game theorists Lloyd S. Shapley and Martin Shubik's power measure.[3] As such, Penrose's place in the narrow literature on the theory of voting hardly seems related to ideas about the regularities governing the spread of pathological ideas.[4] This is unfortunate because, as the present paper argues, Penrose's results in the sphere of

[3] Today, the Penrose index is known as the Penrose-Banzhaf index in honour of the lawyer John F. Banzhaf III's (1965) independent construction of an identical measure twenty years after Penrose. Banzhaf wrote in the context of the 'one person, one vote' court-ordered redistricting revolution in the United States of the progressive 1960s Warren Court era. See Ansolabehere and Snyder (2008) for a detailed account of this 'political equality' revolution and for an empirical analysis of its tangible, policy-wise, significance.

[4] The 1962 typescript, "Pathology of Group Behaviour", mostly borrows from Penrose's 1952 book, *On the Objective Study of Crowd Behaviour*.

voting have their roots in his work on the spread of such pathological ideas. That is to say, his work on equal political representation and on the measurement of voting power was motivated by a concern for much wider and more pernicious phenomena: the outbreak and large-scale propagation of panics, the ideological susceptibility of populations, the escalation of military conflict and the successful installation of authoritarian regimes, the concentration of social control and the type of hierarchical institutions that enable such concentration. More precisely, Penrose's power index was constructed as a *rigorous* way of measuring the social control that a small elite group can sway over a large population while his square root law was meant to capture the ease with which such control is yielded as the population varies in size.

The first, and major, aim of this paper is thus to transpose Penrose's contributions—and, particularly, his famous voting power index—from the peripheral context of voting to the theoretical core of the study of social behaviour. This is important for a number of reasons. First, some of the fiercest *methodological* criticism levied at the Penrose index and, particularly, its underlying assumptions becomes much less justified when placed in the historical context of Penrose's concern with the laws of group behaviour. Indeed, as the first two sections argue, these assumptions were meant to capture descriptive facts about the kind of social groups—importantly, not groups of standard voters—Penrose was concerned with.

Even more importantly, however, recovering the historical context of Penrose's contributions allows us to shift attention from a conception of power grounded in individual abilities to an understanding of power as a *relational* concept. Penrose's interest in the social conditions that allow for the concentration of social control stems from an approach that we might today call structuralist. Economics has been slow in incorporating structural concerns when it comes to the measurement of power—in part, due to the dominant role of game theory and its underlying methodological

individualism. Yet interest in structural issues has been resurging and, as the opening epigraphs attest, ideas that Penrose grappled with more than fifty years ago are gaining new force. To take advantage of these ideas, we need to do justice to the context in which they originated. The bulk of the text is devoted to this task. Before that, however, the next section introduces the place of Penrose's power index, and the criticism it faces, in the contemporary theory of voting power measurement.

THE MEASUREMENT OF VOTING POWER

The common intuition motivating the theory of voting power measurement is simple: voting power "is not directly proportional to the number of votes one casts" (Lucas, 1983, p. 189). Thus, rather than relying on the absolute or relative number of votes a committee member has, one should instead look at her *power*. That is to say, her ability to determine, and thus be decisive or pivotal to, the outcome of a vote. As an illustration, consider a committee of three voters who can cast the following number of votes: 80, 19 and 1, respectively. Suppose further that it takes unanimity, i.e. the approval of all three voters and hence a total of 100 votes, to pass a proposal. Then, despite the large voting weight discrepancies, each committee member is equally powerful: each has the same probability of determining the outcome of the vote.[5]

[5] The explicit distinction between a voter's power and the number of votes she can cast can be traced back to a seminal article by the lawyer John F. Banzhaf III: "In almost all cases weighted voting does not do the one thing which both its supporters and opponents assume that it does: weighted voting does not allocate voting power among legislators in proportion to the population each represents because *voting power is not proportional to the number of votes a legislator may*

Now, there is a key assumption underlying this observation: namely, that of indifference or random voting. The assumption says that no voter has an expressed preference for either the passage or defeat of the bill and that each member of the committee is thus equally likely to vote 'yea' or 'nay'.[6] Crucially, both the Penrose-Banzhaf index and its main rival, the Shapley-Shubik index, assume indifference or random voting.[7] As an empirical claim, the assumption is clearly contentious if not outright wrong. To wit, it ignores "systematic patterns of voting that appear in all voting systems" (Gelman, Katz & Bafumi, 2004, p. 672). Surely, the critics claim, if one wanted to say something about the probability of swinging a vote and if that probability depended on the possible voting configurations in a system, then one has to develop "more complex stochastic processes [.

cast" (1965, p. 318; emphasis in the original). Implicitly using the same intuition, the other founding paper of the discipline—that of game theorists Lloyd Shapley and Martin Shubik—spoke of the *value* of a vote determined by its power: "Put in crude economic terms, the above implies that if votes of senators were for sale, it might be worthwhile buying forty-nine of them, but the market value of the fiftieth (to the same customer) would be zero" (1954, pp. 787–788).

[6] To see how this might make a difference, suppose that the bill in the present example is jointly proposed by voters one and two who thus always support it. The final outcome is then completely determined by the last single-vote member.

[7] There are slight axiomatic and probabilistic differences between the Penrose-Banzhaf index and the Shapley-Shubik index which are well known today. These are not significant for the present argument but see Dubey and Shapley (1979) for the axiomatic differences and Straffin (1988) for the probabilistic differences.

. .] that allow for correlations and unequal probabilities" (Gelman, Katz & Tuerlinckx, 2002, p. 424).

Today the answers to this empirical challenge come in a number of forms. First, there is the argument from *ignorance* or *the principle of insufficient reason*: if we have no information about the "personality and interests" of a voter and "the nature of the bill to be voted upon", then we should assign equal probabilities to her voting either way (Felsenthal & Machover, 1998, p. 38). This, however, begs the question as the distinct bite of the criticism is that we *do* in fact have such relevant information. A more convincing reply is based on a conceptual analysis of what constitutes power: as power is a dispositional concept, it "concerns what players may be *able to do*, not the actions they *may* or *do* take" (Braham & Holler, 2005a, p. 139; emphasis in the original).[8] Power, in other words, is an ability, and when measuring abilities one should abstract away from any preference—we should, that is, assume indifference.[9] A third response relies crucially on the

[8] Braham and Holler (2005a) is part of a prolonged debate on the role of preferences in measuring power. See also Napel and Widgrén (2005), Braham and Holler (2005b), Schmidtchen and Steunenberg (2014) and the 1999 symposium in the *Journal of Theoretical Politics*, *11*(3).

[9] The reply invokes Morriss' *exercise fallacy*—namely, that *having* power differs from *exercising* power (1987, pp. 15–18)—and is related to a distinction between *power* and *success*. That is to say, between making a difference to and agreeing with a voting outcome (Laruelle & Valenciano, 2005). Curiously, Penrose was well aware of this distinction. An unpublished co-authored typescript—a much expanded version of Penrose's famous 1946 article—includes a long discussion of the *satisfaction* a voting bloc may obtain (Penrose & Beall, 1946). These sections were omitted from subsequent publications.

distinction between *a priori* and *a posteriori* power. Random voting models based on the assumption of indifference are concerned with *a priori* power: the ability to make a difference that stems from the "rules themselves" rather than the "particular personalities and political interests present in particular voting environments" (Roth, 1988, p. 9). *A posteriori* (or strategic) power, on the other hand, takes into account all relevant, or at least quantifiable, personal and institutional information, including preferences, cross-player political affinities and positional asymmetries such as special agenda-setting roles.[10] On this account, measures of *a priori* power, such as the Penrose-Banzhaf and the Shapley-Shubik indices, are concerned with the design and evaluation of institutions, and such analyses are properly conducted behind a *veil of ignorance*.

Which of these answers, if any, did Penrose have in mind when he made the assumption of indifference? The answer is that they all sit uneasily with Penrose's own writings. To start with, Penrose's main concern was with populations of voters who have equal votes (namely, one) rather than different voting weights as in the preceding example—an example typical of the contemporary voting power literature. Furthermore, instead of the single *asymmetric* players that populate weighted voting games, and simple games more broadly,[11] we find in Penrose's writings a

[10] This way of distinguishing between *a priori* and *a posteriori* power is not without problems. Strictly speaking, both types of power measures assume some underlying distribution of preferences and hence pattern of behaviour. For this criticism—and for an alternative distinction between *ex ante* and *ex post* measures—see Napel and Widgrén (2004), and Laruelle and Valenciano (2005).

[11] In the domain of cooperative game theory, voting power is normally measured with the use of simple games, i.e. cooperative games whose characteristic function takes a binary value. Weighted

homogeneous indifferent crowd swayed by a resolute person who "knows his mind" (1952, p. 6). Most of the contemporary voting power literature is couched in the language of cooperative game theory and translating Penrose in this language is not without reason. His population of different-sized blocs—if the latter are assumed to vote in unison—can naturally dabble as a committee of asymmetric players in a weighted game. Even then, however, the bulk of Penrose's general arithmetic theory—the power index and the square root law—is restricted to a population of only *two* blocs (namely, a majority and a minority).[12] Given that, in the theory of games, two-player weighted, or simple, games furnish few or no surprising results in the context of voting, it is fair to suppose that Penrose's interest lay elsewhere.

Second, there is no indication that Penrose's indifference assumption relies on a conceptual commitment to viewing power as an ability or a purposeful abstraction from all information that does not concern the 'rules themselves'. On the contrary, there is every reason to believe that Penrose would have been sympathetic to the empirical challenge. He is careful to note, for example, that the assumption of random voting is not unreasonable—*in the context of voting*—as a statistical approximation (1961, p. 546), suggesting an affinity to the *principle of insufficient reason*.[13] The conclusion to be drawn from this is that for Penrose indifference was a descriptive fact,

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voting games are a subset of simple games. For a thorough treatment, which, however, is not necessary for the present discussion, see Maschler, Solan and Zamir (2013).

[12] With the notable exception, as we will see, of an analysis of hierarchical structures.

[13] Penrose also considered deviations from the random voting model in his discussion of 'bias' (1945, pp. 5–6; Penrose & Beall, 1946, pp. 23–31). The latter was, however, omitted from his famous 1946 article.

capturing the fundamental features of a *different* set of situations: only in a second step was it applied to the context of voting assemblies. Indeed, Penrose says as much himself pointing out that the language of "parliamentary procedure" had been chosen "for the sake of convenience" (Penrose & Beall, 1946, p. 43) as in no other field "do appropriate terms seem to exist" (ibid., p. 2). Thus, in order to appreciate the importance of his power index and the square root law, and how reasonable—or unreasonable—an assumption indifference is, we need to revisit Penrose's general theory of crowd behaviour.

AN OBJECTIVE THEORY OF CROWD BEHAVIOUR

> [O]bedience is but a state of inculcated indifference.
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> —Penrose and Beall (1946, p. 40)

It is impossible to separate Penrose's interest in the transmission of abnormal ideas from his main study of human genetics, learning disabilities and mental disorder. Penrose's most prominent work—summarised in his first book in 1933 and continuously re-appraised over the next forty years (1972)—concerned the analysis of the (biological *and* environmental) determinants of mental deficiency.[14] Two principles, which he carried over to the study of crowd behaviour and majority voting, underlay his approach in this work: the belief that statistical data on the distribution of hereditary traits and other socio-biological features are key to elucidating the causes of mental defects and the insistence on avoiding an exclusive focus on genetics both with respect to the study and the treatment of mental deficiencies. Indeed, Penrose's belief in both 'nature' and 'nurture',

[14] He is best remembered for his study of the effect of paternal age and chromosome composition on the heredity of Down's syndrome—see Harris (1974, pp. 6–10) for an overview.

articulated in a 1955 lecture at the National Children's Home,[15] testifies to his widely known distrust of eugenics.[16] The aversion was not just a reaction to sentiments following the Second World War (although references to the rise of National Socialism abound in his texts). Rather the belief in the priority of environment was an active commitment that permeated both his life and work: from his first association with the Psychologists' Peace Society in the 1930s to the cofounding in 1951 of its successor, the Medical Association for the Prevention of War (MAPW).[17]

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[15] This lecture echoed his earlier writings on crowd behaviour: "it seems likely that predisposition to accept one idea rather than another may be determined partly by inborn factors both chemical and structural. [. . .] It is not to be supposed that inborn differences between members of groups of people are likely to be major factors in most examples of crowd behaviour. Far more significant is previous experience, in the form of mode of life, education and general knowledge" (1952, pp. 63–64).

Curiously enough, Penrose's own scientific peace activism can be traced back to a dual (both hereditary and environmental) ancestry: his great-aunt, throughout her life, led a world Peace Society (MacDonald 1972); he was also born in a Quaker family and remained a life-long pacifist.

[16] Nine years after a return from Ontario to London, Penrose managed to change the title of the journal *Annals of Eugenics* to *Annals of Human Genetics*. Another nine years later and two years before his retirement, in 1963, he succeeded in re-naming the Galton Professorship from Galton Professorship of Eugenics to Galton Professorship of Human Genetics.

[17] Penrose's first foray into peace research is telling of his subsequent conception of the nature of war and the appropriate tools for its prevention. Inspired by a 1930s exchange between Albert Einstein and Sigmund Freud—published in a rare 1933 pamphlet, *Why War?*, and preserved in a

The MAPW was just a node in a vast network of peace research institutes that spanned mostly North America, Europe and Asia during the Cold War.[18] In his influential 1961 manifesto *The Bridge of Reason*, Norman Alcock—the co-founder of the Canadian Peace Research Institute whose Board of Directors counted names such as Kenneth Boulding and Canada's future prime minister Pierre Trudeau, among others—summed up the rationale behind the movement in his

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collection of Einstein's peace writings prefaced by Bertrand Russell (Nathan & Norden, 1968, pp. 187–203; see also Paret, 2005)—the Psychologists' Peace Society, much like Penrose himself, viewed war as a psychological problem (MacDonald, 1972; Penrose 1962b). The logical inference, summed up in the opening words of UNESCO's constitution, is that "since wars begin in the minds of men it is in the minds of men that [the defences of] peace must be constructed" (Penrose 1962b). In other words, the aversion of war, and violence more broadly, cannot take beliefs and preferences as given. On the contrary, the very solution is in the efforts to change them.

During two years spent in Vienna, from 1922 to 1923, Penrose had met Freud and even undergone a few sessions of psychoanalysis (Smith, 1999, pp. 17–20). Although his interest in psychoanalysis would eventually wane, the psychological view of abnormal crowd behaviour as a *mental disease* (1952, p. 4) would continue to bear the early marks of Freud's influence (see also Penrose 1925).

[18] The forerunner of these institutes was undoubtedly the Center for Research on Conflict Resolution (CRCR) at the University of Michigan whose pivotal figure, Kenneth Boulding, gave in 1963 a lecture at the ULC on the peace research movement (Boulding, 1962). For the wider history of this movement, centred on the CRCR and its subsequent influence, see Erickson et al. (2013) and Erickson (2015).

concept of the 'critical few'.[19] A bridge between a world of war and a world of peace was possible, according to Alcock, and that bridge was scientific research into and popularisation of knowledge of the causes of war and the means to their alleviation. A closely-knit network of a 'tipping point' number of scientists around the world was to serve this very function, paving the way to a future without violence (Eckhardt, 1983). Unsurprisingly, as a small cog in this network, Penrose saw "[e]ducation, in the widest sense of spread of human knowledge" (1952, p. 65), as the strongest antidote to the occurrence of conflict. In fact, education was not just a key part of Penrose's conclusion but also of the premise of his theory of group behaviour where the indifferent crowd had centre stage.

The indifferent crowd

> When the twenty years just past come to be looked back upon from the

[19] Alcock's pamphlet was among the titles in a small book haul—donated by Penrose's colleague at the Galton Laboratory, Cedric A. B. Smith—which it was hoped would become the library of a future institute at UCL. The other titles were Lewis Richardson's *The Statistics of Deadly Quarrels* and *Arms and Insecurity*, Theodore Lentz's *Towards a Science of Peace*, Anatol Rapoport's *Fights, Games, and Debates*, Penrose's own *On the Objective Study of Crowd Behaviour* and a complete run of the *Journal of Conflict Resolution* (Smith, 1962; Erickson, 2015, p. 224).

Although the institute was never established formally, the group around Smith and Penrose had made UCL an active peace research centre. A year before Boulding, Alcock himself paid a visit, at which Penrose was present, outlining the goals and administration of the Canadian Institute and seeking to join efforts with similar enterprises in the United Kingdom (Hoyte, 1962).

> distant future, it is probable that their chief claim to interest will be that they saw the birth of the science of abnormal psychology.
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> — Trotter (1921 [1916], p. 56)

Wilfred Trotter's foresight might not have been perfect but it had a lasting influence on Penrose's writings on group psychology. The very first page of his *Study of Crowd Behaviour* is unequivocal about Penrose's intellectual allies: the controversial father of crowd psychology in *fin de siècle* France, Gustave Le Bon;[20] UCL's own Wilfred Trotter, whose concept of the 'herd instinct' bridged sociology and psychology in pre-war Britain; and Freud himself. Yet while Penrose's theory, steeped in the belief that the "individual acts differently as a member of a group from the way he acts as an individual" (1952, p. 2), could in this sense be counted among familiar works of the crowd psychology movement, the similarities end in the details.[21] Echoing Victorian fears of the irrational and indecent mob, Le Bon and Trotter epitomised a shared democratic distrust in a call for educating a new elite which it was hoped would "guide the ship of state through the 'era of crowds'" (Nye, 1975, p. 78). Penrose's commitment to education 'in the widest sense', on the

[20] And, by some accounts, the "notorious racist and intellectual servitor of the French military class" (Rieff, 1965 [1960], p. 230).

[21] Robert Nye's *Origins of Crowd Psychology* (1975) is a masterful narrative of the turn-of-the-century movement with Le Bon its main protagonist. See also Nye (1973) and van Ginneken (1985) for more on the French and Soffer (1969) for more on the British context.

other hand, was accompanied by an equally considerable expenditure of effort in the design of an equitable world government.[22]

A more crucial point of disagreement, however, was the nature of the twentieth-century 'crowd'. The authors of the crowd psychology movement saw it as an *unconscious* entity—a

[22] Inspired by, and practically applied to, the recently established United Nations. In 1950, acknowledging an admission "by the United Nations Association and by Mr. Stalin that the present system of voting in the United Nations assembly is unfair", Penrose mused over the possibility—and impossibility—of a world without national borders: "It might be supposed ideally that these divisions of mankind ought to be neglected and that votes should be awarded to artificially equal constituencies all over the world. This has already been proposed by the Crusade for World Government. As things are actually organised, however, differently sized groups do exist, which act as units by virtue of their own separately elected representative governments" (1950, pp. 1–2).

Penrose's solution was to apportion the United Nations assembly according to his now famous square root law. That is to say, the allocation of votes to representatives in the assembly such that a representative's number of votes is proportional to the square root of the population of his or her country. Under the assumptions of the random voting model, such an allocation would give the citizens of each country an equal amount of power (Penrose, 1946, 1950, 1952, 1961). Contemporary expositions of Penrose's square root law are in Felsenthal and Machover (1998, pp. 63–78) and Fielding and Liebeck (1975). Gelman, Katz and Tuerlinckx (2002) is a critical appraisal of the significance of the random voting assumption.

'mass' of non-rational instincts—sweeping through the established order of conservative values.[23] As Le Bon put it: "The substitution of the unconscious action of crowds for the conscious activity of individuals is one of the principal characteristics of the present age" (1896 [1895], p. v).

Similarly, Trotter's 'gregarious' herd was driven by "the passions of the pack" (1921 [1916], p. 115) as it—much like in Freud—succumbed to the authority of a "strong and expert personality" (ibid., pp. 115–116). Thus, the crowd *qua* crowd for these authors was a non-rational mass fed by appetitive rather than cognitive states.

Penrose's crowd was of a different sort. Defined simply as "a number of people who have the same idea" (1952, p. 2), its primitive was the individual rather than the mass.[24] The view taken here had two key implications. The first, to which we shall return shortly, was the importance of the *numerical size* of the group. The second was the conception of crowd behaviour as inherently

[23] As Nye (1993, p. 688) writes, where Marx saw the empowerment of "a disenfranchised and oppressed class", Le Bon saw the "tottering and disappearing" of "ancient beliefs", "while the old pillars of society are giving way one by one" and "the crowd is the only force that nothing menaces" (1896 [1895], p. xv).

[24] "The *fundamental unit* can be defined as a person characterized by possessing a given idea or potential reaction pattern. [. . .] In the present treatment, an attempt will be made to build on the basis of the simple concept of a number of people who have the same idea. I propose to examine some theoretical consequences of such an assumption and further to collect some observations in numerical terms, such as are in general use in epidemiology, about the spread and decay of ideas in groups and crowds" (Penrose, 1952, p. 2; my emphasis).

rational swayed by ideas and beliefs rather than inborn instincts.[25] For Penrose, the psychology of the indifferent crowd was thus the psychology of the indifferent individual, whose indifference followed not from the ruling of an inner many-headed beast but from a failure of critical rationality.

One might think that these roots of Penrose's type of indifference in the domain of rationality position him, even if indirectly,[26] in the tradition of the public choice school that was emerging at the time in the United States. And it is true that, like Penrose, the school of Duncan Black, James Buchanan, Gordon Tullock, Anthony Downs and Mancur Olson, among others, was concerned with the collective action of groups of *rational* individuals, rather than of a non-rational mass. However, there is a difference between 'crowd behaviour' as Penrose understood it and the 'group behaviour' or 'collective action' of the public choice authors.[27]

A cornerstone of the work of these authors was the process by which, in Penrose's words, resolute individuals who know their mind arrive at a collective choice. Or, in the language of rational choice theory, the process by which a group of individuals with *given* complete preferences aggregate these preferences so as to arrive at a collective choice. Penrose departed from this tradition in two important respects. First, he was exclusively concerned with situations in which *not everyone* knows their mind. That is to say, his analysis applied to conditions under which

[25] "One definite property of the *ideas* discussed here is that they are supposed to be acquired and not inborn characteristics of the individual" (Penrose, 1952, p. 67; emphasis in the original).

[26] The work of the early public choice authors does not refer to Penrose's writings on voting or crowd behaviour. It is thus doubtful that these authors were acquainted with Penrose's results.

[27] Selected histories of the public choice school are in Mitchell (1999), Medema (2000) and Amadae (2003).

failures of critical thinking stall the process of individual reflection necessary for rational choice. Indeed, in his 1962 draft on the "Pathology of Group Behaviour", Penrose listed four "characteristics of groups" that are "sufficiently constant to be referred to as rules or Laws" (1962a, p. 3). The first of these corresponds to the assumption of indifference in Penrose's writings on voting and concerns the peculiar features of individual behaviour as individuals become members of groups:

> The first Law is undoubtedly the paradoxical relationship between the behaviour of the members [of groups] as individual and as group participants, which has already been discussed here. This change of morals with *diminution of critical faculties* is of fundamental importance in the study of war prevention because nations, through their political and military leaders, tend to act in an infantile manner towards one another.[28]

This 'diminution of critical faculties', according to Penrose, is one of the "unfavourable aspects" (1962a, p. 1) of the way individual behaviour is modified as individuals join groups:

> From the moral and intellectual point of view, the change [of behaviour] has both favourable and unfavourable aspects. The unfavourable alternatives are weakening of moral judgment and critical sense. The process has been described as regression towards infantile patterns of reaction, characterised by intolerance, violence, suggestibility and unquestioning allegiance to the leader or to the authority of the rules of the group. The favourable modifications are those concerned with self sacrifice, loyalty and comradship.[29]

[28] Penrose (1962a, p. 3; my emphasis).

[29] Penrose (1962a, p. 1).

The second difference between Penrose's understanding of 'group behaviour' and that of the public choice authors is that Penrose had no interest in the *process* that leads to the idea shared by the members of a crowd. He had, in other words, no interest in the problem of aggregation—a problem that lies at the very heart of the public choice school and other traditions inspired by Kenneth Arrow's social choice work.

In effect, we could say that Penrose's notion of indifference is grounded in-between the ideas of Le Bon, Trotter et al. and those of the rational choice school with respect to the concept of a group. Penrose's 'inculcated indifference' was not that of the unconscious mass, at the one extreme, or of the fully rational collection of individuals, at the other. Rather, it was a susceptibility stemming from a failure in one's critical faculties: a failure on a large social scale due to a lack of education, misinformation and distrust.

It is worth pausing here to draw an important distinction. The idea of indifference underlying Penrose's random voting model should be distinguished from that of apathy. Penrose's indifferent voters are not the apathetic voters of the late twentieth century portrayed by political scientist Peter Mair (2013). Instead of former partisans sapped of enthusiasm who refuse to be swayed, they are undecided individuals only too ready to be swayed. In this sense, rather than perfectly inelastic, they are perfectly elastic to any political innovation calling them to action.

Motivated by the Cold War polarisation of power and escalating fears of a nuclear conflict, the use of propaganda in the East *and* West, the 'population question' and its relevance to poverty, famine and the ethical codes of the medical profession, Penrose's random voting model was a

simplified tool designed to elucidate all these problems.[30] Imagine, with Penrose, a population split into two groups: a "free-acting or unorganized" indifferent majority and a "resolute or controlled" minority (1945, p. 1). The former is, characteristically, "divided in opinion in a random manner" (1952, p. 6) while the latter is bound by a decision already taken in advance. This description is clearly idealised yet it was taken as sufficiently similar to conditions enabling the propagation of crazes, panics and other more or less serious outbreaks.[31] The two crucial conditions were lack of information and critical reflection, and isolation:

[30] The short summary of the second conference organised in 1962 by the MAPW on the subject of 'The Pathogenesis of War' is a compendium of various topics recurring throughout the Association's documents and Penrose's writings. A reprint of the article, originally published in the medical weekly *The Lancet*, is in MAPW (1962). See Bashford (2007) for a general history of the interwar debates on the 'population question'.

[31] A small-town bank run in the US, the mass religious migration of the Doukhobors in 1902 Canada and Orson Welles' 1938 *War of the Worlds* radio debacle are among the examples of panics in "politics, national enthusiasms, commercial advertising, art, war and religion" discussed by Penrose (1952, p. 4). Curiously, economic theory was counted among the more benign forms of "mental disturbances". Unfortunately, Penrose never elaborated on this point.

In his 1962 manuscript on the "Pathology of Group Behaviour", Penrose singled out the two-block assumption as a separate 'law of group behaviour' which says that "groups tend to develop in opposing pairs"; that is to say, "every organized crowd seems to produce opposition" (1962a, p. 3).

> Strictly, it is only when a fresh idea is presented to people, who have not had previous opportunities for its consideration, that an approximation is reached to the formal conditions outlined here.[32]

The indifferent crowd was thus the uninformed and ignorant crowd. For Penrose, the danger which the twentieth century crowd presented was not the uncontrollable outburst of primordial mass instincts but the lack of reflection in large swathes of "people isolated by ignorance" (1952, p. 68). It was for the purpose of quantifying this danger that he employed his now famous voting power index.

The effect of the size of the crowd

> A perennial feature of the structure of society is the mechanics by which large groups of people are controlled in some degree by smaller groups.
>
> —Penrose and Beall (1946, p. 1)[33]

Given a large indifferent group of size n_I and a smaller resolute group of size n_R, how effective is the latter in controlling the former? Penrose's answer was couched in the vocabulary of voting but

[32] Penrose (1952, p. 12). Also: "One of the most important factors predisposing to disruptive epidemics is isolation" (Penrose, 1952, p. 65). Isolated individuals—and groups—were accounted for by the assumption of *independent* voting, or action more generally. In a similar vein a decade later, Penrose's colleague at UCL and one of the fathers of evolutionary game theory, John Maynard Smith, pointed out that a necessary condition for natural selection to occur at the level of the group is the full or partial isolation of that group (1964, p. 1145).

[33] In the typescript, the word 'distinctive' is manually substituted for 'perennial'.

it can easily be translated in more general terms. Suppose that 'a fresh idea is presented to people' such that there are two possible ways of reacting to it. Such an idea could be a policy proposal put to a 'yea' or 'nay' (referendum) vote, a novel ideology or piece of propaganda that can be embraced or rejected, or an impeding threat which people can give in to or not. Suppose further that the n_I irresolute people decide *independently* to take either action with equal probability one half.[34] In other words, the members of the indifferent camp are "equally likely to be persuaded" to react in either direction (Fielding & Liebeck, 1975, p. 252). Finally, suppose that for the idea to become prevalent in the population, it needs to be endorsed by at least a majority of the people.[35]

Given these assumptions, how easily can an unwavering group—such as a political party spreading a piece of propaganda—sway the population in its preferred direction? Very easily, Penrose replied, and the larger the crowd is, the easier it becomes. To see this, notice that the resolute camp can "alter the attitude of the whole group" (1952, p. 6) when the group is divided in opinion. That is to say, when half of it endorses and the other half opposes the idea. The probability of this

[34] More precisely, if S is the random variable denoting the number of people supporting one of the two actions, then S is governed by the binomial distribution with a probability of success equal to $1/2$.

[35] This threshold follows from the restriction of majority voting where the quota is one half. Some ideas have clearly lower and others higher thresholds: the tipping point for a fire panic in a crowded theatre is perhaps not more than one person while adopting a conventional code of conduct might require near unanimity.

happening is precisely Penrose's power index.[36] Now, given the assumption of random action, the probability of the resolute camp swaying the entire population is inversely proportional to the square root of the number of people in the indifferent crowd, i.e. the power of the n_R bloc varies together with $n_R/\sqrt{n_I}$. Practically, this means that a resolute group of three wields the same power over a crowd of nine as does a group of ten over a crowd of a hundred (namely, about 70%) while a bloc of 300 is effectively a dictatorship—or rather a "tyranny" (Penrose & Beall, 1946, p. 38)—in a community of 10,000 people.

Today, this relation between the size of a crowd and the degree with which it can be controlled is known as the square root law: a principle used for the *fair*—or "equitable" (Penrose, 1966)—allocation of votes in representative systems. This was indeed one of the problems—and a very natural one—to which Penrose applied his power measure and the "square root deal" (1950) that follows from it.[37] The domains which originally motivated him, however, as we have seen, had considerably higher stakes: the propagation of fear among vast numbers of people; the ideological infection of the very fabric of a country, its population; and not least of all, the successful establishment of tyrannical or authoritarian forms of government. To appreciate the thrust of the last

[36] In the 1946 typescript, he calls it the *value* rather than the power of a vote and ascribes its significance to the fact that it furnishes a measure of interference with *freedom* (Penrose & Beall, 1946, p. 17).

[37] As previously mentioned, the particular voting body was the assembly of the newly established United Nations (Penrose, 1946; 1950; 1952, pp. 44–46 and 72–74; 1961; 1966).

point, we need to revisit a seldom broached[38] extension of the square root law governing the power of resolute blocs: namely, hierarchical control.

Hierarchical control

The problem of small minorities controlling large majorities was further exacerbated, according to Penrose, in hierarchical governance structures.[39] Suppose, as before, that there are two groups in a society: a resolute minority and an irresolute majority. Now zero in on the minority bloc and suppose that it too contains an unwavering sub-bloc structured in the same fashion. In the resulting *matryoshka*-like system each successive resolute group is controlled by its (smaller) predecessor in the hierarchy. Thus, while the power of the "central authority would be diminished at each step" (Penrose, 1952, p. 9), its control over a large indifferent population would far surpass that which a direct system allows:

> A central group of 15, for example, would normally have only one chance in one thousand of influencing the decisions of a population of 100 million. But with four intermediate stratified blocs of 25, 69, 529 and 30,000 people respectively the resolute

[38] With the notable exception of Fielding and Liebeck (1975). The omission might be attributed to the fact that the topic furnished only a brief two-paragraph mention in the famous 1946 article. It is somewhat more extensively discussed in Penrose's 1952 book. Its most detailed analysis, however, is undoubtedly in the 1946 typescript co-authored with Geoffrey Beall.

[39] The terminology throughout his writings is not entirely consistent. Such systems are sometimes referred to as "hierarchical" (Penrose, 1946; 1952, p. 9) or "stratified" (Penrose, 1952, p. 9), and other times as systems of "blocs within blocs" (Penrose & Beall, 1946).

> group of 15 would retain control of 99 per cent. of the decisions of 100 million random voters.[40]

The nested-blocs system allowed Penrose to explain how resolute groups emerge in the first place.[41] More importantly, however, it elucidated the conditions favourable to extreme forms of power consolidation such as those in "authoritarian and semi-authoritarian systems—*e.g.*, in military, ecclesiastical and industrial organizations" (Penrose, 1946, p. 54). Three of these conditions proved particularly fruitful in understanding the establishment of tyrannies (Penrose & Beall, 1946, pp. 38–41). First, as hierarchies trump more diffuse systems in terms of efficiency of control, the emergence of a "central oligarchy" must proceed alongside the "breaking up [of] even the apparently most harmless clubs and societies" (Penrose, 1946, p. 54).[42] And, second, as efficiency improves with the size of a country, "the recent immense increase in population numbers", Penrose and Beall (1946, p. 39) concluded, must have greatly facilitated the spread of dictatorial forms of government.[43] Finally, Penrose did not shy away from admitting that the effect of the size of the crowd on the latter's susceptibility to control crucially depends on the assumption of indifference. Yet for him this only buttressed the fact that the harder dictatorial regimes tried to suppress communication and spread ignorance of political issues (Penrose & Beall, 1946, p. 39; Penrose, 1946,

[40] Penrose (1952, p. 10).

[41] And clarifies his reference to the indifferent camp as "unorganized" and to the resolute group as "controlled" (1945, p. 1).

[42] Penrose's example here is Germany in the era of National Socialism.

[43] A point attributed to José Ortega y Gasset's 1930 volume *The Revolt of the Masses* that, however, was part of the larger 'population problem' which animated the peace research group at UCL.

p. 54), the closer they would approximate the model's ideal conditions and the gravity of its implications. In a word, for Penrose, obedience was indeed "inculcated indifference"—both "arithmetically" (Penrose & Beall, 1946, p. 40) and empirically.

From today's vantage point, Penrose's hierarchical system—much like his power index and the square root law—might naturally speak the language of simple games and their composition which were being developed at the same time by Lloyd Shapley (1954) at the RAND Corporation. Its spirit, however, is much closer to a more recent subset of the theory of networks spurred by Sah and Stiglitz' (1986) work on hierarchies and polyarchies.[44] Although Sah and Stiglitz' initial interest in these structures lay in their information diffusion properties, they have subsequently started to raise a question reminiscing issues entertained by Penrose half a century ago: namely, how does power depend on the structure governing a society or organisation?[45]

[44] A distinction carefully drawn in Penrose and Beall's 1946 typescript—but omitted from later publications and hence, one can only speculate, due to Beall rather than Penrose— offers a striking parallel. One should distinguish, they argued, between "blocs within blocs" on the one hand—which has so far been referred to as 'hierarchies'—and actual "hierarchies" on the other (1946, p. 33). The former are vertical structures where lower groups are "overborne" by their superiors (corresponding to Sah and Stiglitz' hierarchical architectures) while the latter are conglomerations of separate blocs-within-blocs systems (corresponding to mixed polyarchies *of* hierarchies in Sah and Stiglitz).

[45] For recent papers bridging Sah and Stiglitz' work, on the one hand, and Penrose's, on the other, see van den Brink and Steffen (2008, 2012).

For Penrose, this was a question important enough to deserve a separate ‘Law’ on its own. In the 1962 manuscript on pathological behaviour, the fourth and final law-like characteristic of group behaviour is:

> [. . .] the tendency for the development of a hierarchical system of subclasses within the membership [of the whole group]. A small resolute section can dominate the rest, but an even smaller section can itself control the dominant class and thus control the whole mass. [. . .] In such circumstances the control of a group can be very efficient indeed in that a very small body of men or perhaps even one man only can dictate behaviour to a whole nation.[46]

There is thus a gravity to Penrose’s power index and the larger theory in which it is embedded—including the square root law and the regularities of hierarchical structures—a gravity that is lacking from the other (voting) power indices such as the Banzhaf and the Shapley-Shubik measures. It concerns not just the abilities of individuals, such as political representatives, to effect policy outcomes but also the abilities of individuals to affect the lives of large groups of other *individuals* through subjugation and social control. Penrose’s power index and his other results must thus be more properly understood not just as a subclass of the theory of voting but also of a larger interdisciplinary programme cutting through issues of politics, sociology, anthropology, psychology, philosophy and statistics, to name a few.

[46] Penrose (1962a, p. 4).

CONCLUSION

The central claim of this paper is that it is important to recognise the much wider historical and intellectual context of Penrose's major claim to prominence today—his voting power index and his square root law. Without diminishing the import or usefulness of their applications today, these formal tools were not just tools of the theory of majority voting but of a larger programme on the regularities governing social behaviour. When evaluating the assumptions underlying these tools then, it is appropriate to be sensitive to their original purpose. If we do that, as the paper has argued, we will see Penrose's voting power index transforming into a much more ambitious measure of social control, the gravity of whose consolidation is described by his square root law.

In fact, while today Penrose's index is taken as a dispositional measure— namely, of the *ability* to effect outcomes—it was originally much closer in spirit to *relational* conceptions of power. This might be unsurprising as, prior to the 1980s, to analyse power—predominantly within sociology, anthropology and political science—meant to analyse *power over* relations rather than *power to* abilities. In the late 1980s, the philosopher Peter Morriss turned the tide by claiming that '[e]verything that needs to be said about power can be said using the idea of the capacity to effect outcomes' (1987, p. 34).[47] Perhaps so but then Penrose's measured warning of the danger of large ignorant crowds controlled by an unwavering few should be easily translatable in more contemporary terms. As the opening epigraphs attest, the time for this might have finally come.

[47] In other words, to have power over someone or to control someone means nothing more than to have the ability to determine outcomes involving that person.

REFERENCES

Amadae, S. M. (2003). *Rationalizing capitalist democracy: The Cold War origins of rational choice liberalism*. Chicago, IL: University of Chicago Press.

Ansolabehere, S., & Snyder Jr., J. M. (2008). *The end of inequality: One person, one vote and the transformation of American politics*. New York, NY: W. W. Norton & Co.

Banzhaf III, J. F. (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, *19*(2), 317–343.

Bashford, A. (2007). Nation, empire, globe: The spaces of population debate in the interwar years. *Comparative Studies in Society and History*, *49*(1), 170–201.

Boulding, K. (1962). The Peace Research Movement and the contribution of science to stable peace. In 'Peace Research Group', L. S. Penrose Papers, PENROSE/2/6/8/4, University College London Library Special Collections.

Braham, M., & Holler, M. J. (2005a). The impossibility of a preference-based power index. *Journal of Theoretical Politics*, *17*(1), 137–157.

Braham, M., & Holler, M. J. (2005b). Power and preferences again: A reply to Napel and Widgrén. *Journal of Theoretical Politics*, *17*(3), 389–395.

Dubey, P., & Shapley, L. S. (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, *4*(2), 99–131.

Eckhardt, W. (1983). Pioneers of Peace Research VI: Norman Z. Alcock: Apostle of cycles and builder of bridges. *International Interactions: Empirical and Theoretical Research in International Relations*, *10*(2), 247–267.

Erickson, P. (2015). *The world the game theorists made*. Chicago, IL: The University of Chicago Press.

Erickson, P., Klein, J. L., Daston, L., Lemov, R., Sturm, T., & Gordin, M. D. (2013). *How reason almost lost its mind: The strange career of Cold War rationality*. Chicago, IL: The University of Chicago Press.

Felsenthal, D., & Machover, M. (1998). *The measurement of voting power: Theory and practice, problems and paradoxes*. Cheltenham: Edward Elgar.

Fielding, G., & Liebeck, H. (1975). Voting structures and the square root law. *British Journal of Political Science*, *5*(2), 249–256.

Gelman, A., Katz, J. N., & Bafumi, J. (2004). Standard voting power indexes do not work: An empirical analysis. *British Journal of Political Science*, *34*(4), 657–674.

Gelman, A., Katz, J. N., & Tuerlinckx, F. (2002). The mathematics and statistics of voting power. *Statistical Science*, *17*(4), 420–435.

Harris, H. (1974). Lionel Sharples Penrose (1898–1972). *Journal of Medical Genetics*, *11*(1), 1–24.

Hoyte, W. N. (1962). Letter to Penrose with a report of a meeting on the occasion of Norman Alcock's visit. In 'Peace Research Group', L. S. Penrose Papers, PENROSE/2/6/8/7, University College London Library Special Collections.

Laruelle, A., & Valenciano, F. (2005). Assessing success and decisiveness in voting situations. *Social Choice and Welfare*, *24*(1), 171–197.

Le Bon, G. (1896 [1895]). *The crowd: The study of the popular mind*. New York, NY: Macmillan.

Lucas, W. F. (1983). Measuring power in weighted voting systems. In S. J. Brams, W. F. Lucas, & P. D. Straffin, Jr. (Eds.), *Political and related models* (pp. 183–238). New York, NY: Springer-Verlag.

MacDonald, N. (1972). Lionel S. Penrose – 1898–1972. *Medical Association for the Prevention of War Proceedings*, *2*(5), 117–122. In 'Obituaries', L. S. Penrose Papers, PEN-ROSE/1/20/5, University College London Library Special Collections.

Mair, P. (2013). *Ruling the void: The hollowing-out of Western democracy*. New York, NY: Verso.

MAPW (1962). The pathogenesis of war. In 'Medical Association for the Prevention of War', L. S. Penrose Papers, PENROSE/2/6/3, University College London Library Special Collections.

Maschler, M., Solan, E., & Zamir, S. (2013). *Game theory*. Cambridge: Cambridge University Press.

Medema, S. G. (2000). "Related disciplines": The professionalization of public choice analysis. *History of Political Economy*, *32*(Suppl 1), 289–324.

Misak, C. (2020). *Frank Ramsey: A sheer excess of powers*. Oxford: Oxford University Press.

Mitchell, W. C. (1999). Political science and public choice: 1950–70. *Public Choice*, *98*(3–4), 237–249.

Morriss, P. (1987). *Power: A philosophical analysis*. Manchester: Manchester University Press.

Napel, S., & Widgrén, M. (2004). Power measurement as sensitivity analysis: A unified approach. *Journal of Theoretical Politics*, *16*(4), 517–538.

Napel, S., & Widgrén, M. (2005). The possibility of a preference-based power index. *Journal of Theoretical Politics*, *17*(3), 377–387.

Nathan, O., & Norden, H. (Eds.). (1968). *Einstein on peace*. New York, NY: Schocken Books.

Nye, R. A. (1973). Two paths to a psychology of social action: Gustave LeBon and Georges Sorel. *The Journal of Modern History*, *45*(3), 411–438.

Nye, R. A. (1975). *The origins of crowd psychology: Gustave LeBon and the crisis of mass democracy in the Third Republic*. London: SAGE Publications.

Nye, R. A. (1993). The rise and fall of the eugenics empire: Recent perspectives on the impact of biomedical thought in modern society. *The Historical Journal*, *36*(3), 687–700.

Paret, P. (2005). Einstein and Freud's pamphlet *Why War?*. *Historically Speaking*, *6*(6), 14–19.

Penrose, L. S. (1925). The relation of the pleasure-pain principle of Freud to the question of growth. In L. S. Penrose Papers, PENROSE/2/1/8, University College London Library Special Collections.

Penrose, L. S. (1933). *Mental defect*. London: Sidgwick and Jackson.

Penrose, L. S. (1945). The control of groups by resolute minorities. In 'Group Control', L. S. Penrose Papers, PENROSE/2/7/2/2, University College London Library Special Collections.

Penrose, L. S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, *109*(1), 53–57.

Penrose, L. S. (1950). International voting (The square root deal). In L. S. Penrose Papers, PENROSE/2/7/3, University College London Library Special Collections.

Penrose, L. S. (1952). *On the objective study of crowd behaviour*. London: H. K. Lewis & Co. Ltd.

Penrose, L. S. (1955). Heredity and environment in human affairs. *The 1955 Convocation Lecture of the National Children's Home*. London: National Children's Home.

Penrose, L. S. (1958). Mechanics of self-reproduction. *Annals of Human Genetics*, *23*(1), 59–72.

Penrose, L. S. (1961). Some statistical problems of majority voting. *New Scientist*, *9*(224), 546–547.

Penrose, L. S. (1962a). Pathology of group behaviour. In L. S. Penrose Papers, PENROSE/2/6/9, University College London Library Special Collections.

Penrose, L. S. (1962b). War as a behavioural problem. In L. S. Penrose Papers, PENROSE/2/6/10, University College London Library Special Collections.

Penrose, L. S. (1966). Equitable voting in the United Nations. In 'Statistics of Elections', L. S. Penrose Papers, PENROSE/2/7/1/5, University College London Library Special Collections.

Penrose, L. S. (1972). *Biology of mental defect* (4th ed.). London: Sidgwick and Jackson.

Penrose, L. S., & Beall, G. (1946). The arithmetic of group control. In 'Group Control', L. S. Penrose Papers, PENROSE/2/7/2/4, University College London Library Special Collections.

Penrose, L. S., & Penrose, R. (1958). Impossible objects: A special type of visual illusion. *British Journal of Psychology*, *49*(1), 31–33.

Rieff, P. (1965 [1960]). *Freud: The mind of the moralist*. London: University Paperbacks/Methuen.

Roth, A. E. (1988). Introduction to the Shapley value. In A. E. Roth (Ed.), *The Shapley value: Essays in honor of Lloyd S. Shapley* (pp. 1–27). Cambridge, New York, Melbourne: Cambridge University Press.

Sah, R. K., & Stiglitz, J. E. (1986). The architecture of economic systems: Hierarchies and polyarchies. *The American Economic Review*, *76*(4), 716–727.

Schmidtchen, D., & Steunenberg, B. (2014). On the possibility of a preference-based power index: The strategic power index revisited. In R. Fara, D. Leech, & M. Salles (Eds.), *Voting power*

and procedures: Essays in honour of Dan Felsenthal and Moshé Machover (pp. 259–286). Cham, Heidelberg, New York, Dordrecht, London: Springer.

Shapley, L. S. (1954). Simple games: An outline of the descriptive theory. The RAND Corporation: Research Memorandum RM-1384.

Shapley, L. S., & Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *The American Political Science Review*, *48*(3), 787–792.

Smith, C. A. B. (1962). Letter to the members of the "Peace Research Group". In 'Peace Research Group', L. S. Penrose Papers, PENROSE/2/6/8/3, University College London Library Special Collections.

Smith, J. M. (1964). Group selection and kin selection. *Nature*, *201*, 1145–1147.

Smith, M. (1999). *Lionel Sharples Penrose: A biography*. Colchester, Essex: Michael Smith.

Soffer, R. A. (1969). New elitism: Social psychology in Prewar England. *Journal of British Studies*, *8*(2), 111–140.

Straffin, Jr., P. D. (1988). The Shapley-Shubik and Banzhaf power indices as probabilities. In A. E. Roth (Ed.), *The Shapley value: Essays in honor of Lloyd S. Shapley* (pp. 71–81). Cambridge, New York, Melbourne: Cambridge University Press.

Thaler, R. H., & Sunstein, C. R. (2009). *Nudge: Improving decisions about health, wealth and happiness*. London: Penguin Books.

Trotter, W. (1921 [1916]). *Instincts of the herd in peace and war*. London: T. Fisher Unwin Ltd.

van den Brink, R., & Steffen, F. (2008). Positional power in hierarchies. In M. Braham, & F. Steffen (Eds.), *Power, freedom, and voting: Essays in honour of Manfred J. Holler* (pp. 57–81). Berlin, Heidelberg: Springer-Verlag.

van den Brink, R., & Steffen, F. (2012). Axiomatizations of a positional power score and measure for hierarchies. *Public Choice*, *151*(3–4), 757–787.

van Ginneken, J. (1985). The 1895 debate on the origins of crowd psychology. *Journal of the History of the Behavioral Sciences*, *21*(4), 375–382.

von Neumann, J. (1951). The general and logical theory of automata. In L. A. Jeffress (Ed.), *Cerebral mechanisms in behavior: The Hixon symposium* (pp. 1–41). New York, NY: John Wiley and Sons.